On Some Problems of M. Z. Nashed on Outer Inverses

Boris Shekhtman
University of California, Riverside
Department of Mathematics
Riverside, California 92521

Submitted by R. S. Varga

ABSTRACT

While not every linear operator on a Banach space has a generalized inner inverse, the situation for outer inverses is different. Every operator on a Banach space has an outer inverse. That answers one problem posed by M. Z. Nashed.

Let X and Y be Banach spaces, and let $L(X,Y)$ be the space of continuous linear operators from X into Y.

Let $A \in L(X,Y)$. An operator $B \in L(Y,X)$ is called an *inner inverse* for A if

$$ABA = A,$$

and an *outer inverse* for A if

$$BAB = B$$

It is well know that $A \in L(X,Y)$ has an inner inverse iff $\ker A$ and $\operatorname{range} A$ are complemented in X and Y, respectively.

It is also easy to see that every $A \in L(X,Y)$ has a trivial outer inverse: $B = 0$.

In [1, problems (a), (b), (c)], Professor Nashed asked for a criterion in terms of A for the existence of a nontrivial bounded outer inverse.

The following simple observation provides a solution to all three problems.

THEOREM 1. *Let $F : X \to Y$ be an arbitrary homogeneous map. F has a nontrivial outer inverse in $L(Y,X)$ iff $F \neq 0$.*

52 Vanderbilt Ave., New York, NY 10017 0024-3795/86/$3.50

Proof. Since $F \neq 0$, there exists $x_0 \in X$ so that

$$0 \neq Fx_0 \in Y.$$

By the Hahn-Banach theorem we can find a linear continuous functional $\varphi \in Y^*$ so that

$$\varphi(Fx_0) = 1.$$

Now the outer inverse is given by

$$By = \varphi(y)x_0, \qquad y \in Y. \tag{1}$$

Indeed

$$BFBy = B(\varphi(y)F(x_0)) = \varphi(y)\varphi(F(x_0))x_0 = By. \qquad \blacksquare$$

REMARK 1. If x_0 is a unique (up to a constant) element in X with $F(x_0) \neq 0$, then the outer inverse (1) is unique up to the choice of the functional φ.

Theorem 1 has the following simple modification:

THEOREM 2. *Let $A \in L(X,Y)$, and let*

$$\dim \operatorname{range} A \geq n \geq 0. \tag{2}$$

Then, for every $m \leqslant n$, there exists an outer inverse to A with an m-dimensional range.

Proof. By assumption, we can pick m linearly independent elements $x_1, \dots, x_m \in X$ so that the elements $Ax_1, \dots, Ax_m$ are linearly independent.Then, the Hahn-Banach theorem furnishes us with m functionals $\varphi_1, \dots, \varphi_m \in Y^*$ biorthogonal to $(Ax_1, \dots, Ax_m)$, i.e., $\varphi_j(Ax_i) = \delta_{ij}$. Now, an outer inverse B is given by

$$B(y) = \sum_{i=1}^{n} \varphi_i(y)x_i. \qquad \blacksquare \tag{3}$$

REMARK 2. From the definition of the outer inverse, it is easy to see that the construction (3) is the only one possible in the following sense:

(a) If $\dim \operatorname{range} A = n$, then an outer inverse B has to have at most an n-dimensional range.

(b) Since BA has to be a projection, the only way we can choose functional φ_i is biorthogonal to (Ax_i).

REMARK 3. Theorem 2 shows that the questions on the existence of outer inverses are more challenging if we ask for the existence of outer inverse with such properties as "being removed from the finite-dimensional operators."

Here is a simple example of such a statement.

THEOREM 3. *An operator $A \in L(X, Y)$ has an outer inverse with an infinite-dimensional range iff there exists a closed infinite-dimensional subspace $X_0 \subset X$ such that $Y := AX_0$ is complemented in Y and*

$$X_0 \cap \ker A = \{0\}. \tag{4}$$

Proof. Let $A_0 = A|X_0 : X_0 \to Y_0$. Then

$$\ker A_0 = \{0\} \qquad \operatorname{range} A_0 = Y_0.$$

Hence there exists an inverse $A_0^{-1} : Y_0 \to X_0$. Let P_0 denOte the projection from Y onto Y_0. Define B by

$$B_y = A_0^{-1} P_0 y.$$

We claim that B is an outer inverse to A.

Indeed,

$$BABy = BA\left[A_0^{-1}(P_0 y)\right] = BP_0 y = A_0^{-1} P_0 P_0 y = A_0^{-1} P_0 y = By.$$

The second equality follows from the fact that $P_0 y \in Y_0$; the fourth from $P_0^2 = P_0$. Conversely, let B be an outer inverse to A with an infinite-dimensional range: X_0. Then, it is obvious that $\operatorname{range} B \cap \ker A = \{0\}$. Clearly, AB is a projection onto a subspace $Y_0 := AX_0 = ABY$. ■

REMARK 4. In particular, if $\ker A$ and $\operatorname{range} A$ are complemented, we can choose X_0 so that

$$\ker A \oplus X_0 = X.$$

The last theorem can be used to construct an example of an operator $A \in L(X, Y)$ with an infinite-dimensional range that does not have an outer inverse with an infinite-dimensional range. Indeed, let $X = l_1$, $Y = c_0$, and A be the inclusion mapping of X into Y. Then for A to have an outer inverse we have to find an infinite-dimensional subspace Y_0 in $\operatorname{range} A$ which is complemented in c_0 and isomorphic to a subspace of l_1. This is impossible, since (cf. [2]) every such subspace $Y_0 \subset c_0$ must be isomorphic to c_0 and hence cannot be isomorphic to the subspace of l_1.

Professor G. Gierz has noticed that the range of A cannot contain any infinite-dimensional *closed* subspaces of c_0.

The next example deals with the case when Y is reflexive and was suggested to me by the referee.

Let $A: l_1 \to l_p$ be the identity, where $1 < p < \infty$. If A had an outer inverse, then there would exist a $Y_0 \subset l_p$, complemented, such that Y_0 is isomorphic to a subspace of l_1. Take $\{y_n\} \subset Y_0$ with $\|y_n\| = 1$, $y_n \to 0$ weakly in Y_0. Then $y_n \to 0$ weakly in l_1, and thus $\|y_n\| \to 0$. Contradiction.

The author is pleased to acknowledge many suggestions by the referees. Professor M. Z. Nashed has recently informed me that similar results were obtained independently in M. Z. Nashed "On generalized inverses and operator ranges" ISNM 60, pp. 85–96.

REFERENCES

1 M. Z. Nashed, Best approximation problems arising from generalized inverse operator theory, in *Approximation Theory III* (E. W. Cheney, Ed), 1980, pp. 667–673.

2 A. Pelczynski, Projections in certain Banach spaces, *Studia Math.* 19:209–228 (1960)

Received 2 February 1985; revised 26 March 1985